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16 Saving honey

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Problem

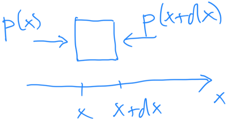
When rotating a rod coated with a viscous liquid (e.g. honey), under certain conditions the liquid will stop draining. Investigate this phenomenon.

Literature

- German site (GYPT 2022): video
  `https://www.gypt.org/aufgaben/16-saving-honey.html`
- Canadian site:
  `https://stemfellowship.org/iypt-references/problem16/`
- first analytical discussion:
  H. K. Moffatt, *Behaviour of a viscous film on the outer surface of a rotating cylinder*, J. Mec. 16, 651 (1977)
- more recent numerical analysis:
  P. L. Evans, L. W. Schwartz, and R. V. Roy, *Three-dimensional solutions for coating flow on a rotating horizontal cylinder: Theory and experiment*, Phys. Fluids 17, 072102 (2005)
- see also:
  Newtonian liquids, properties of honey
  Navier-Stokes equations, capillary length, Rayleigh-Taylor instability
  ...

Acceleration of the liquid: 1. pressure

acceleration $\vec{f}$ if the pressure field $p(\vec{x})$ is non-constant, $\rho =$ density



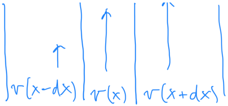
$$F = -\left[p(x+dx) - p(x)\right] \cdot S$$

$$f = \frac{F}{\rho\, dx \cdot S} = -\frac{p(x+dx) - p(x)}{dx}$$

$$\vec{f} = -\frac{1}{\rho} \vec{\nabla} p$$

Acceleration of the liquid: 2. viscous forces

acceleration if the velocity field is non-constant, $\nu =$ kinematic viscosity



internal friction:

$$\frac{\partial v}{\partial t} = \text{const} \cdot \left[v(x+dx) + v(x-dx) - 2v(x) \right]$$

$$v(x \pm dx) = v(x) \pm \frac{\partial v}{\partial x} \cdot dx + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \cdot dx^2$$

$$\frac{\partial v}{\partial t} = \text{const} \cdot \frac{\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial \vec{v}}{\partial t} = \nu \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{v}}$$

Acceleration of the liquid: 3. inertial effects

if the velocity fields depends on both time and space coordinates:

$$v(t+dt, x+dx) = v(t,x) + \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot \frac{dx}{dt}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot \frac{dx}{dt}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \underbrace{\left(\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}\right)}_{\vec{v}\cdot\vec{\nabla}} v$$

$$\boxed{\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}}$$

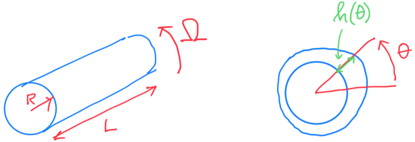
so-called total (hydrodynamic) derivative

Navier-Stokes equations for honey

- Navier-Stokes equations:

$$\frac{\partial \vec{v}}{\partial t} = \vec{g} - \frac{1}{\rho}\vec{\nabla} p + \nu(\vec{\nabla}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla})\vec{v}$$

- special case: fluid at rest, $\vec{\nabla} = 0$; then $\vec{g} = \frac{1}{\rho}\vec{\nabla} p$
- our case:



- 8 parameters: R, L, Ω, h, g, σ, ρ, ν
  to begin with, neglect L (two-dimensional approximation)

Dimensionless parameters

- 7 parameters: R, Ω, h, g, σ, ρ, ν
- three dimensionful parameters, e.g. R, g, and ρ
- four dimensionless parameters, e.g.:

$$
\begin{aligned}
\epsilon &= \frac{\text{thickness}}{\text{radius}} = \frac{h}{R} \to h \\
W &= \frac{\text{inertia}}{\text{gravity}} = \frac{R\Omega^2}{g} \to \Omega \\
N &= \frac{\text{viscous force}}{\text{gravity}} = \frac{\nu R\Omega/h^2}{g} \to \nu \\
S &= \frac{\text{capillary length}}{\text{radius}} = \frac{\sqrt{\sigma/(\rho g)}}{R} \to \sigma
\end{aligned}
$$

- various regimes!

Stationary solution of Moffatt

- assume:
  $\epsilon \ll 1$ (thin films), $W \ll 1$ (inertia negligible),
  $S \ll 1$ (surface tension negligible)
- current flows dominantly along the cylinder, component $u = u(r, \theta)$
- differential equation (only viscous and gravity terms):

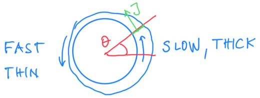
$$\nu \frac{\partial^2 u}{\partial r^2} = g \cos\theta$$

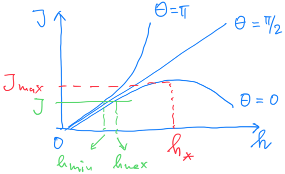
- boundary conditions for $r = R$ (inner) and $r = R + h(\theta)$ (outer surface):

$$u(R) = R\Omega, \qquad \frac{\partial u}{\partial R} = 0$$

- solution:

$$u(r, \theta) = R\Omega + \frac{g\cos\theta}{2\nu} f(r), \qquad f(r) = r^2 - 2(R + h)r + R(R + 2h)$$





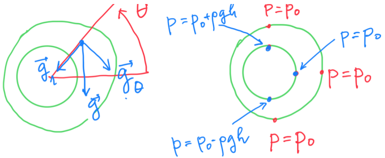
- total flow at angle θ

$$J(\theta) = \int_R^{R+h} dr u(r,\theta) = R\Omega h - \frac{gh^3}{3\nu}\cos\theta$$

- since $J(\theta) =$const, this defines the shape $h = h(\theta)$
- average thickness h_0 possible only for $\Omega > \Omega_c$, where

$$\boxed{\Omega_c = \left(\frac{2\pi}{4.443}\right)^2 \frac{gh_0^2}{\nu R}}$$

Explanation of Moffatt's solution



- gravity force $\vec{g} = \vec{g}_r + \vec{g}_\theta$
- tangential part $g_\theta = -g \cos\theta$ compensated by viscous forces
- radial part $g_r = -g \sin\theta$ compensated by pressure, therefore

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -g \sin\theta$$

- (strange) solution for pressure $p(r, \theta)$:

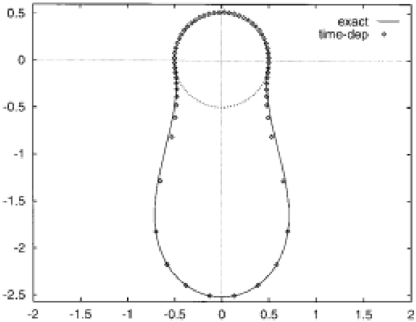
$$p(r, \theta) = p_0 + \rho g (R + h - r) \sin\theta$$

- pressure at surface: $p(R + h, \theta) = p_0 =$ atmospheric pressure (equilibrium)

Opposite extreme: static cylinder $\Omega = 0$

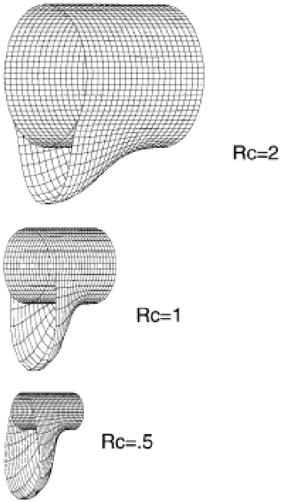
Weidner et al., J. Colloid and Interface Science 187, 243 (1997)

- surface tension can not be ignored any more
- length scale set by capillary length $\ell_c = \sqrt{\sigma/(\rho g)}$
- two-dimensional approximation: droplet hangs under the cylinder (a ridge at $\theta = -\pi/2$ is formed)

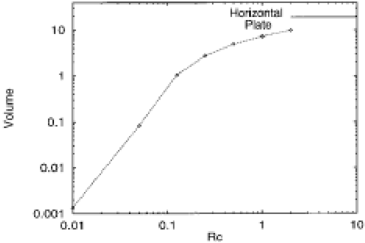


Static cylinder $\Omega = 0$, axial instability

Weidner et al., J. Colloid and Interface Science 187, 243 (1997)



- the ridge below the cylinder is unstable
- drops of size $\sim \ell_c$ form along the cylinder
- volume of the drops (units of ℓ_c^3):



Suggestions

- choose a highly viscous Newtonian liquid and a cylinder of radius R; determine how h_0 (or mass of the liquid) depends on the frequency Ω; compare with theory; does an optimal Ω exist?
- try to estimate the material parameters σ, ρ, ν; identify the relevant dimensionless parameters
- do the results depend on the initial conditions of the experiment? how do the flow instabilities influence the results?
- try to change other parameters besides Ω (e.g., temperature, radius R, length L, shape of the rod,...)